[6] Wu, J. C. and Thompson, J. F., "Numerical Solution of Time-Dependent Incompressible Navier-Stokes Equations Using an Integro-Differential Formulation," *Computers and Fluids*, Vol. 1, No. 2, 1973, pp. 197–215.

[7] Chorin, A. J., "Numerical Study of Slightly Viscous Flow," *Journal of Fluid Mechanics*, Vol. 57, 1973, pp. 785–796.

[8] Lugt, H. J. and Haussling, H. J., "Laminar Flow Past a Flat Plate at Various Angles of Attack," *Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics*, Springer-Verlag, New York, 1970, pp. 78–83; also Rept. 3748, Feb. 1972, Naval Ship Research and Development Center, Bethesda, Md.

[9] Batchelor, G. K., "The Source of Vorticity in Motions Generated from Rest," *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1970, pp. 277–282.

[10] Karamcheti, K., "Elements of Thin Airfoil Theory," *Principles of Ideal-Fluid Aerodynamics*, Wiley, New York, 1966, pp. 492–517.

[11] Gentry, R. A., Martin, R. E., and Daly, B. J., "An Eulerian Differencing Method for Unsteady Compressible Flow Problems," *Journal of Computational Physics*, Vol. 1, 1966, pp. 87–118.

[12] Schmall, R. A., "Unsteady Viscous Flow Past a Lifting Plate," Ph.D. thesis, 1974, Univ. of Arizona, Tucson, Ariz.

[13] Pao, Y.-H. and Daughty, R. J., "Time-Dependent Viscous Incompressible Flow Past a Finite Flat Plate," Rept. D1-82-0822, Jan. 1969, Boeing Scientific Research Labs., Seattle, Wash.

NOVEMBER 1974 AIAA JOURNAL VOL. 12, NO. 11

When is Hamilton's Principle an Extremum Principle?

DONALD R. SMITH*
University of California, San Diego, Calif.
AND
C. V. SMITH JR.†
Georgia Institute of Technology, Atlanta, Ga.

For a dynamic system, the action I for motion from time t_0 to time t_1 can be written as $I = \int_{t_0}^{t_1}(T-V)\,dt$, where T is the kinetic energy and V is the potential energy. In many references to Hamilton's principle, particularly in the engineering literature, it is stated or implied that the true motion of the system will give the action an extreme value, usually stated to be a minimum value. In this Paper it is first demonstrated by very simple examples that Hamilton's principle is not, in general, an extremum principle. Then a relatively elementary and direct proof, which does not require sufficiency theory from the calculus of variations, is presented which shows that for certain discrete linear systems the action is always minimized over short time intervals; and a precise characterization is given for the maximum length of the time interval over which the action is guaranteed to be minimized by the true solution. Finally, it is shown that for continuous systems the action is never minimized by the true solution.

Introduction

FOR the restricted but commonly occurring problem of determining the motion of a conservative system, Hamilton's principle states that the true motion between time t_0 and time t_1 will be such as to satisfy the following equation:

$$\delta \int_{t_0}^{t_1} (T-V)\,dt = 0 \tag{1}$$

where T = kinetic energy of the system and V = potential energy of the system. This equation is presented in texts from mathematics,[1–4] physics,[5] structural mechanics,[6] and structural dynamics[7]; and in most references it is stated that the true solution not only makes the quantity

$$I = \int_{t_0}^{t_1} (T-V)\,dt \tag{2}$$

(called the action of the motion) stationary but also gives I an extremum value, usually stated to be a minimum value. For example, in Ref. 2, it is stated that the actual motion "makes the functional $\int_{t_0}^{t_1}(T-U)\,dt$ a minimum." In Ref. 7 it is said, "The stationary value is actually a minimum." Or in Ref. 5, "...out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2, it will actually travel along that path for which the integral...is an extremum, whether a minimum or maximum...."

Only a relatively few references,[1,3,4] generally in the literature of the calculus of variations, give the more precise statement of Hamilton's principle in which it is recognized that the action is possibly minimized only over short intervals of time. Reference 3 contains a brief treatment of one example in which an alternate form of the action integral is not minimized by a true solution over long intervals of time. (However, for that example if the action is defined as given in Eq. (2), then any true solution minimizes the action for all intervals of time.)

In this paper it will be demonstrated by both example and proof that in general Hamilton's principle for conservative systems is not an extremum principle. This will be accomplished by considering the free vibration of linearly elastic systems. Then it will be shown that for discrete linear systems, "in fact, the action is always minimized over short time intervals..." as stated in Ref. 1, and a precise characterization will be given for the maximum length of the time interval over which the action can be guaranteed to be minimized by the true solution. Finally, for continuous systems it will be shown that the action is never minimized by the true solution.

It should be noted that these results, while evidently not well-known in engineering literature, are available in Ref. 4, where

Received January 28, 1974; revision received June 24, 1974.
Index categories: Structural Dynamic Analysis.

* Associate Professor, Department of Mathematics.
† Associate Professor, School of Aerospace Engineering. Member AIAA.

their development is based upon the general sufficiency theory from the calculus of variations. The purpose of this paper is to first use very simple examples for demonstration and then to present a direct and relatively elementary proof which does not require the notion of conjugate points and other concepts from the general sufficiency theory.

Examples

The first system to be considered is the simple single degree of freedom harmonic oscillator consisting of a mass m and a linear spring of stiffness k, with position of the mass denoted by u. For free vibration, the action is given by

$$I[u] = \frac{1}{2}\int_{t_0}^{t_1} [m\dot{u}^2 - ku^2]\,dt \tag{3}$$

[In Eq. (3) and the following, $(\dot{\ }) = d/dt$.] The true motion satisfies the differential equation

$$m\ddot{u} + ku = 0 \tag{4}$$

and if the example initial conditions are chosen as

$$\text{at} \quad t=0: \quad u=0, \quad \dot{u}=v_0 \tag{5}$$

then the exact solution is

$$u_e = \frac{v_0}{\omega}\sin\omega t \tag{6}$$

where $\omega^2 = k/m$. This is the true motion of the mass, which gives the action a stationary value with respect to all possible motions which pass through the same point at $t = t_0 = 0$ and the same point at an arbitrary value of time $t = t_1$.

Now, a family of possible motions which pass through the same points at $t = 0$ and $t = \bar{t}$ can be constructed as follows, where $\bar{t}$ is any specified time and C is an arbitrary constant:

$$\tilde{u} = \frac{v_0}{\omega}\sin\omega t + C\frac{1}{\bar{t}^2}(t^2 - t\bar{t}) \tag{7}$$

Clearly, at both $t = 0$ and $t = \bar{t}$, $\tilde{u} = u_e$; this is all that is required for $\tilde{u}$ to be an admissible function for comparison of action values with the exact solution. It can be shown by direct substitution into Eq. (3), with limits of integration of $t_0 = 0$ and $t_1 = \bar{t}$, that

$$\Delta I \equiv I[\tilde{u}] - I[u_e] = \tfrac{1}{2}mC^2\frac{1}{30\bar{t}}[10 - \omega^2\bar{t}^2] \tag{8}$$

Obviously, $\Delta I < 0$ if $\bar{t}^2 > 10/\omega^2$. This is sufficient to demonstrate that $I[u_e]$ is not a relative mimimum for all time.

The possible motions in Eq. (7) do not satisfy all initial conditions in Eq. (5). Another family of motions which do satisfy all initial conditions is given by

$$\tilde{u} = \frac{v_0}{\omega}\sin\omega t + C\frac{1}{\bar{t}^3}(t^3 - t^2\bar{t}) \tag{9}$$

For this group of motions

$$\Delta I = \tfrac{1}{2}mC^2\frac{1}{105\bar{t}}(14 - \omega^2\bar{t}^2) \tag{10}$$

and once again $\Delta I < 0$ for sufficiently large $\bar{t}^2$, which means that $I[u_e]$ is not a relative minimum.

It is possible to devise more elaborate families of motion. For example,

$$\tilde{u} = \frac{v_0}{\omega}\sin\omega t + C\frac{1}{\bar{t}^4}[t^4 - t^3(\bar{t}_1 + \bar{t}_2) + t^2\bar{t}_1\bar{t}_2] \tag{11}$$

will satisfy all initial conditions plus $\tilde{u} = u_e$ at $t = \bar{t}_1$ and $t = \bar{t}_2$. However, the result is always the same—the magnitude of ΔI is a function of the time interval, and for sufficiently large values of the time interval, $\Delta I < 0$ and $I[u_e]$ is not minimum.

The second example represents a continuous system. Consider a simply supported uniform beam of length L, bending stiffness EI, and mass per unit length ρA. Let x be the axial coordinate and w be the transverse displacement. For free-bending vibration in one plane, neglecting shear deformation and rotary inertia, the action is given by

$$I[w] = \frac{1}{2}\int_{t_0}^{t_1}\int_0^L \left[\rho A\dot{w}^2 - EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx\,dt \tag{12}$$

The true motion satisfies the differential equation

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\ddot{w} = 0 \tag{13}$$

and if the initial conditions are

$$\text{at} \quad t=0: \quad w=0 \quad \text{and} \quad \dot{w} = v_0\sin\frac{\pi x}{L} \tag{14}$$

then the exact solution is given by

$$w_e = \frac{v_0}{\omega}\sin\frac{\pi x}{L}\sin\omega t \tag{15}$$

where $\omega^2 = (\pi/L)^4 EI/\rho A$.

With a continuous system there are the possibilities of varying either the spatial dependence or the temporal dependence or both when developing families of possible motion. Consider first a family of the form

$$\tilde{w} = \sin\frac{\pi x}{L}\left[\frac{v_0}{\omega}\sin\omega t + C\frac{1}{\bar{t}^3}(t^3 - t^2\bar{t})\right] \tag{16}$$

which satisfies all boundary conditions and all initial conditions, plus $\tilde{w} = w_e$ at $t = t_0 = 0$ and $t = \bar{t}$. Substitution into Eq. (12) with time limits of $t_0 = 0$ and $t_1 = \bar{t}$ gives

$$\Delta I = \tfrac{1}{2}C^2\rho A\frac{L}{210\bar{t}}[14 - \omega^2\bar{t}^2] \tag{17}$$

Since the spatial dependence for $\tilde{w}$ is the same as for w_e, the system has essentially been reduced exactly to a one degree of freedom system; and the limit on $\bar{t}^2$ is the same as shown in Eq. (10) for the harmonic oscillator.

Next assume that possible motions are of the form

$$\tilde{w} = \frac{v_0}{\omega}\sin\omega t\left[\sin\frac{\pi x}{L} + C\frac{1}{L^4}(x^4 - 2x^3L + xL^3)\right] \tag{18}$$

which satisfy all boundary conditions, do not satisfy all initial conditions of Eq. (14), and possess the property that $\tilde{w} = w_e$ at $t = n\pi/\omega$, where $n = 0, 1, 2, \ldots$. Substitution into Eq. (12) with time limits of $t_0 = 0$ and $t_1 = n\pi/\omega$ gives

$$\Delta I = \tfrac{1}{2}C^2\left(\frac{n\pi}{2\omega}\right){v_0}^2\rho AL\frac{1}{630}\left[31 - \frac{21(144)}{\pi^4}\right] < 0 \tag{19}$$

In this case ΔI is always negative, which shows once again that for a continuous system the action is not minimized by the true motion.

The point has already been made for the continuous system, but there are several other families of motion with differing characteristics which might be mentioned. For example, assume

$$\tilde{w} = w_e + C\frac{1}{L^4}(x^4 - 2x^3L + xL^3)\frac{1}{\bar{t}^3}(t^3 - t^2\bar{t}) \tag{20}$$

which satisfies all boundary conditions, all initial conditions, and $\tilde{w} = w_e$ at $t_0 = 0$ and $t_1 = \bar{t}$. Note that both spatial and temporal dependence have been varied in Eq. (20). It can be shown that based on Eq. (20)

$$\Delta I = \tfrac{1}{2}C^2\rho AL\frac{1}{\bar{t}}\frac{8}{175\pi^4}(13.96 - \omega^2\bar{t}^2) \tag{21}$$

For the continuous system, all types of possible motions considered, Eqs. (16, 18, and 20), have had one feature in common. In every case

$$\tilde{w} = w_e + w^*$$

where w^* denotes the perturbation measured from w_e. This was possible because w_e is known for the simple example problem. In practical applications, however, the exact solution is generally not known; and then it is necessary to determine an approximate solution by some method. As an example of this procedure, assume that an approximate solution for the simply supported beam is taken in the form

$$\tilde{w}(x,t) = \frac{4}{L^4}(xL - x^2)q(t) \tag{22}$$

with $q(t)$ unspecified. Equation (22) does not satisfy the stress-type boundary conditions (at the ends, $d^2w/dx^2 = 0$), but it does satisfy the geometric boundary conditions (at the ends, $w = 0$), which is all that is required for use of Hamilton's principle. Substitution of Eq. (22) into Eq. (12) gives the action as

$$\tilde{I} \equiv I[\tilde{w}] = \tfrac{1}{2}\rho AL\frac{8}{15}\int_{t_0}^{t_1}[\dot{q}^2-\tilde{\omega}^2q^2]\,dt \tag{23}$$

where $\tilde{\omega}^2 = (120/L^4)EI/\rho A$.

Requiring $\delta\tilde{I} = 0$ results in the differential equation

$$\ddot{q}+\tilde{\omega}^2q = 0$$

and if the initial conditions are

$$\text{at} \quad t = 0: \quad \tilde{w} = 0 \quad \text{and} \quad \dot{\tilde{w}} = v_0\frac{4}{L^2}(xL-x^2)$$

then the solution for $q(t)$ is

$$q(t) = 4\frac{v_0}{\tilde{\omega}}\sin\tilde{\omega}t$$

From this it follows that

$$\tilde{w}_e = 4\frac{v_0}{\tilde{\omega}}\frac{1}{L^2}(xL-x^2)\sin\tilde{\omega}t \tag{24}$$

and

$$\tilde{I}_e \equiv I[\tilde{w}_e] = \tfrac{1}{2}\rho AL\frac{8}{15}v_0^2\frac{1}{2\tilde{\omega}}\sin 2\tilde{\omega}t_1 \tag{25}$$

with $t_0 = 0$ and t_1 = any arbitrary value.

It can be shown by substituting Eq. (15) into Eq. (12) that

$$I_e \equiv I[w_e] = \tfrac{1}{2}\rho AL\tfrac{1}{2}v_0^2\frac{1}{2\omega}\sin 2\omega t_1 \tag{26}$$

again for $t_0 = 0$.

Now, even though $\tilde{w}_e$, Eq. (24), and w_e, Eq. (15), do not satisfy the same initial condition on velocity, it is still possible to make a meaningful comparison between $\tilde{I}_e$ and I_e if a time t_1 can be determined for which $\tilde{w}_e(x, t_1) = w_e(x, t_1)$. It can be seen that this is possible only if $\sin\tilde{\omega}t_1 = \sin\omega t_1 = 0$, which is possible (for $t_1 > 0$) only if $\tilde{\omega}$ is a rational multiple of ω. For this particular example, $\tilde{\omega}$ is not a rational multiple of ω, which means that a meaningful comparison between $\tilde{I}_e$ and I_e cannot be made. In other words, it is pointless to ask whether I_e is less than or greater than $\tilde{I}_e$ since there is never a nonzero value of time at which $\tilde{w}_e = w_e$.

The preceding discussion based on Eq. (22) illustrates how Hamilton's principle can be utilized to obtain approximate solutions to continuous systems through what is essentially a Ritz-type assumption. However, the fact that the action is not minimized by the true solution points out a distinction between a static analysis using the principle of minimum potential energy (assumed applicable) and a dynamic analysis using Hamilton's principle. In the static problem, the magnitude of potential energy can often serve as a means of comparing approximate solutions; the lower the potential (therefore closer to the *exact* minimum potential), the better the solution. In dynamic problems, however, the magnitude of the action does not, in general, supply such a "measure of goodness" because the exact solution action is not a minimum value for arbitrary time, as is shown in the next section.

General Discussion

The examples presented above have been quite sufficient to show that Hamilton's principle is not an extremum principle, even though in most cases there was a short time interval from 0 to some limiting value of $\bar{t}$ for which $\Delta I > 0$. However, these values of $\bar{t}$ were based on very restricted perturbations to the exact solution, so that it is not possible to guarantee that the action is minimized with respect to all possible motions passing through the same end points. The purpose of this section is to determine the maximum length of the time interval over which the action can be guaranteed to be minimized by the true solution.

Consider a discrete system with generalized displacements $q_i(t)$, $i = 1, 2, \ldots, N$. The kinetic energy T and the potential energy V can, by assumption, be written in the following forms:

$$T = \tfrac{1}{2}m_{ij}\dot{q}_i\dot{q}_j \tag{27}$$

$$V = Q_iq_i+\tfrac{1}{2}k_{ij}q_iq_j \tag{28}$$

where the mass matrix m_{ij} and the stiffness matrix k_{ij} are symmetric and assumed to be constants. The Q_i represent generalized external forces which might be acting on the system. [In Eqs. (27) and (28) and in what follows, the summation convention is utilized, with the indices summed over all admissible values.]

The action is therefore given by

$$I[q] = \frac{1}{2}\int_{t_0}^{t_1}(m_{ij}\dot{q}_i\dot{q}_j-k_{ij}q_iq_j-Q_iq_i)\,dt \tag{29}$$

Let

$$q_i = \bar{q}_i+q_i^* \tag{30}$$

where q_i^* represents the perturbation measured from $\bar{q}_i$. The only requirement on the q_i^* is that $q_i^*(t_0) = q_i^*(t_1) = 0$. Then

$$\Delta I \equiv I[\bar{q}+q^*]-I[\bar{q}]$$

$$\Delta I = \int_{t_0}^{t_1}(m_{ij}\dot{\bar{q}}_i\dot{q}_i^*-k_{ij}\bar{q}_jq_i^*-Q_iq_i^*)\,dt + \frac{1}{2}\int_{t_0}^{t_1}(m_{ij}\dot{q}_i^*\dot{q}_j^*-k_{ij}q_i^*q_j^*)\,dt \tag{31}$$

If $\bar{q}_i(t)$ is identified as the true motion of the system, then the first integral on the right-hand side of Eq. (31) will vanish due to the stationary behavior of $I[q]$ in the neighborhood of the true solution.

Next let $\tilde{q}_j^\alpha$ represent the α eigenvector which satisfies the following equation[8]

$$\omega_\alpha^2m_{ij}\tilde{q}_j^\alpha = k_{ij}\tilde{q}_j^\alpha \tag{32}$$

where ω_α is the frequency of free vibration in the α mode. The eigenvectors are linearly independent and satisfy the following orthogonality relations:

$$m_{ij}\tilde{q}_i^\alpha\tilde{q}_j^\beta = \begin{Bmatrix}0, & \alpha\neq\beta\\ M_\alpha, & \alpha=\beta\end{Bmatrix} \tag{33}$$

$$k_{ij}\tilde{q}_i^\alpha\tilde{q}_j^\beta = \begin{Bmatrix}0, & \alpha\neq\beta\\ K_\alpha=\omega_\alpha^2M_\alpha, & \alpha=\beta\end{Bmatrix} \tag{34}$$

where M_α is the generalized mass associated with the α mode, and every $M_\alpha > 0$ since m_{ij} is positive definite. The quantity K_α is the generalized stiffness, and the sign of K_α and ω_α^2 depends upon k_{ij}.

Now the completely arbitrary perturbation q_i^* is written as a combination of the eigenvectors, as follows:

$$q_i^*(t) = C_\alpha(t)\tilde{q}_i^\alpha, \quad \text{sum on} \quad \alpha \tag{35}$$

subject to the requirements that $C_\alpha(t_0) = C_\alpha(t_1) = 0$ for every α. Substituting into Eq. (31) (without the first integral) gives

$$\Delta I = \frac{1}{2}\int_{t_0}^{t_1}\sum_\alpha[M_\alpha\dot{C}_\alpha^2-\omega_\alpha^2M_\alpha C_\alpha^2]\,dt. \tag{36}$$

Then interchanging summation and integration and using the inequality (39) given in the Appendix, it follows that

$$\Delta I \geqq \sum_\alpha\tfrac{1}{2}M_\alpha\left[1-\omega_\alpha^2\frac{(t_1-t_0)^2}{\pi^2}\right]\int_{t_0}^{t_1}\dot{C}_\alpha^2\,dt \tag{37}$$

From (37) it can now be concluded that a sufficient condition for $\Delta I > 0$ for nontrivial $\dot{C}_\alpha$, which means that the action I has a relative minimum for the true motion, is

$$(t_1-t_0)^2 < \tau^2 \tag{38}$$

with

$$\tau^2 = \frac{\pi^2}{\max(\omega_\alpha^2)}$$

where $\max(\omega_\alpha^2)$ denotes the maximum of all the positive ω_α^2. If all ω_α^2 are non-positive, then $\Delta I > 0$ for every value of (t_1-t_0).

The condition (38) is not only sufficient but also necessary if the action I has a relative minimum for the true motion. Indeed if the given time interval is sufficiently large so as to violate (38), with

$$(t_1 - t_0)^2 \geqq \pi^2/\omega_{\bar{\alpha}}^2$$

where $\bar{\alpha}$ denotes the eigenvector with maximum positive eigenvalue, then the perturbation can be taken as

$$C_\alpha(t) = \begin{cases} 0 & \text{for} \quad \alpha \neq \bar{\alpha} \\ C \sin[\pi(t-t_0)/(t_1-t_0)] & \text{for} \quad \alpha = \bar{\alpha} \end{cases}$$

for any constant C; and Eq. (36) gives the result

$$\Delta I = \frac{\pi^2 C^2 M_{\bar{\alpha}}}{4(t_1 - t_0)}\left[1 - (t_1 - t_0)^2 \frac{\omega_{\bar{\alpha}}^2}{\pi^2}\right] \leqq 0$$

which contradicts the assumption that I has a relative minimum.

Condition (38) can be given the following physical interpretation. If $\omega_\alpha^2 > 0$ for some value of α, then the system is capable of simple harmonic motion in the associated α eigenvector; and the period of vibration is given by $\tau_\alpha = 2\pi/\omega_\alpha$. Therefore, it can be said that a necessary and sufficient condition for minimum action is that the time interval must be less than one half of the minimum period of free vibration. If the system can not execute simple harmonic motion, then the action is minimized for all time intervals.

Although the preceding discussion was based on a discrete system, it can be made applicable to a continuous system by discretizing in terms of the eigenvectors of free vibration of the continuous system. If this is done, then it can be shown that ΔI will have the form given by Eq. (36), with an infinite sum on α; and (38) will again supply a necessary and sufficient condition for $\Delta I > 0$ for arbitrary perturbations. However, there is no maximum eigenvalue for a continuous system. This means that no matter how small the time interval it will always be possible to find a perturbation which gives $\Delta I < 0$. Therefore, the action is never minimized by the true solution in a continuous system. Of course, if an approximate solution is sought by a Ritz-type assumption, then the infinite degree of freedom continuous system has effectively been replaced by a finite discrete system; and Eq. (38) is valid for the discretized system.

Summary

Since the action, I, of any motion is a function of the time interval, it follows that ΔI (the difference between perturbed motion and true motion actions) will also be a function of the time interval. It has been shown for a discrete system that there is a positive number τ (related simply to the minimum period of free vibration of the system), such that $\Delta I > 0$ for any time interval of length $t_1 - t_0 < \tau$, while $\Delta I < 0$ for certain perturbations if the time interval has length $t_1 - t_0 > \tau$. That is, for a discrete system the action is truly minimized by the exact solution in the interval $t_0 < t < t_1$ if and only if $t_1 - t_0 < \tau$. However, for a continuous system, the action is never minimized, no matter how small the time interval.

Appendix

The following well-known inequality is used above in deriving Eq. (37):

$$\pi^2 \int_{t_0}^{t_1} h(t)^2\, dt \leqq (t_1 - t_0)^2 \int_{t_0}^{t_1} h'(t)^2\, dt \tag{39}$$

where Eq. (39) is valid for all continuously differentiable real-valued functions h on the fixed interval $t_0 \leqq t \leqq t_1$ which vanish at the end points with $h(t_0) = h(t_1) = 0$. The inequality (39) is actually valid for a larger class of functions h subject to milder smoothness conditions. The inequality (39) is sharp since equality occurs for the admissible function $h^*(t) = C \sin[\pi(t-t_0)/(t_1-t_0)]$ for any constant C.

The proof of Eq. (39) follows from Wirtinger's inequality[9,10] by first changing the independent variable to the special case of $t_0 = 0$, $t_1 = \pi$ and then extending the function h to be an odd function on the larger interval $-\pi \leqq t \leqq \pi$ by defining $h(t) = -h(-t)$ for $-\pi \leqq t \leqq 0$. This extended function will satisfy the conditions for validity of Wirtinger's inequality.

Finally, Wirtinger's inequality can be proved by several different methods. The proof can be obtained from the general sufficiency theory of the calculus of variations as in Ref. 4 and Ref. 9. There is a more direct method of proof due to Hurwitz[10] which we prefer since it is based entirely on certain well-known and elementary results from the theory of Fourier series. We refer the interested reader to the literature for the details.

References

[1] Smith, D. R., *Variational Methods in Optimization*, Prentice-Hall, Englewood Cliffs, N.J., 1974, pp. 195–205.

[2] Lebedev, N. N., Skalskaya, I. P., and Uflyand, Y. S., *Problems of Mathematical Physics*, Prentice-Hall, Englewood Cliffs, N.J., 1965, p. 392 (supplement prepared by E. L. Reiss).

[3] Hestenes, M. R., "Elements of the Calculus of Variations," *Modern Mathematics for the Engineer*, edited by Beckenbach, McGraw-Hill, New York, 1956, pp. 75–78.

[4] Gelfand, I. M. and Fomin, S. V., *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, N.J., 1963, pp. 159–162.

[5] Goldstein, H., *Classical Mechanics*, Addison-Wesley, Reading, Mass., 1950, p. 31.

[6] Dym, C. L. and Shames, I. H., *Solid Mechanics: A Variational Approach*, McGraw-Hill, Inc., New York, 1973, p. 353.

[7] Meirovitch, L., *Analytical Methods in Vibrations*, Collier-Macmillan, London, 1967, pp. 42–45.

[8] Noble, B., *Applied Linear Algebra*, Prentice-Hall, Englewood Cliffs, N.J., 1969, pp. 396–398.

[9] Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge University Press, Cambridge, England, 1952, pp. 184–187.

[10] Beckenbach, E. F. and Bellman, R., *Inequalities*, 2nd Revised Printing, Springer-Verlag, New York, 1965, pp. 177–178.